

M304T: Linear Algebra

Canonical Form

Nilpotent transformation

Dr. R. Bhuvaneshwari

BMS College for Women

November 18, 2020

Nilpotent transformation

Let V be an n -dimensional vector space over a field F .

Definition

A scalar $\lambda \in F$ is called a characteristic root (or eigen value) of $T \in A(V)$ if $T - \lambda I$ is singular.

That is, if \exists a non-zero vector $v \in V$ such that

$$vT = \lambda v.$$

v is called characteristic vector of T corresponding to λ .

Note

T can be brought to a diagonal form if and only if there exists a basis for V consists of characteristic vectors of T .

Definition

If $\lambda \in F$ is a characteristic root of T then the null space of $T - \lambda I$ is called eigen space of λ .

Eigen space of $\lambda = \{v \in V : v \neq 0 \& vT = \lambda v\} = \ker(T - \lambda I)$.

Remark

0 is a characteristic root of T if and only if T is not one-one.

Definition

A linear transformation $T \in A(V)$ is called nilpotent if $T^k = 0$ for positive integer k .

(i.e) there exists a positive integer k such that $vT^k = 0$ for all $v \in V$.

Definition

The smallest positive integer k such that $T^k = 0$ is called nilpotent index of T .

\therefore , if T is a nilpotent transformation of index m then $T^m = 0$ but $T^{m-1} \neq 0$. (Note that $T^k \neq 0$ for $k < m$).

Result

All characteristic roots of a nilpotent linear transformation are zero.

Proof: Let T be a nilpotent linear transformation of nilpotent index m . Then $T^m = 0$ but $T^{m-1} \neq 0$. Let α be a characteristic root of T . Then \exists a $v \neq 0$ in V such that

$$vT = \alpha v.$$

Claim: $\alpha = 0$.

$$vT = \alpha v \implies vT^2 = (vT)T = (\alpha v)T = \alpha(vT) = \alpha(\alpha v) = \alpha^2 v.$$

Continuing this way, we get

$$vT^3 = \alpha^3 v, \dots, vT^m = \alpha^m v.$$

Since $T^m = 0$, $vT^m = 0$ and so $\alpha^m v = 0$. Since $v \neq 0$,

$\alpha^m v = 0 \implies \alpha^m = 0$. This implies that $\alpha = 0$.

Result

If $T \in A(V)$ is nilpotent and $\alpha_0 \neq 0$ then

$\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k$ is invertible.

Proof:

First we prove that if S is nilpotent and $\alpha \neq 0$ then $\alpha_0 + S$ is invertible.

Let $S \in A(V)$ is nilpotent. Then $S^m = 0$ for some integer $m > 0$.

Let $\alpha_0 \neq 0$. Then

$$\begin{aligned}
& (\alpha_0 + S) \left(\frac{I}{\alpha_0} - \frac{S}{\alpha_0^2} + \frac{S^2}{\alpha_0^3} + \cdots + (-1)^{m-1} \frac{S^{m-1}}{\alpha_0^m} \right) \\
&= I - \frac{S}{\alpha_0} + \frac{S^2}{\alpha_0^2} - \frac{S^3}{\alpha_0^3} + \cdots + (-1)^{m-1} \frac{S^{m-1}}{\alpha_0^{m-1}} \\
&\quad + \frac{S}{\alpha_0} - \frac{S^2}{\alpha_0^2} + \frac{S^3}{\alpha_0^3} + \cdots + (-1)^{m-2} \frac{S^{m-1}}{\alpha_0^{m-1}} + (-1)^{m-1} \frac{S^m}{\alpha_0^m}
\end{aligned}$$

Since $S^m = 0$ we have

$$(\alpha_0 + S) \left(\frac{I}{\alpha_0} - \frac{S}{\alpha_0^2} + \frac{S^2}{\alpha_0^3} + \cdots + (-1)^{m-1} \frac{S^{m-1}}{\alpha_0^m} \right) = I.$$

Similarly, we can show that

$$\left(\frac{I}{\alpha_0} - \frac{S}{\alpha_0^2} + \frac{S^2}{\alpha_0^3} + \cdots + (-1)^{m-1} \frac{S^{m-1}}{\alpha_0^m} \right) (\alpha_0 I + S) = I$$

Thus $(\alpha_0 I + S)$ is invertible.

Given T is nilpotent. Then $T^m = 0$ and $T^{m-1} \neq 0$

Consider $S = \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k$

Then $S \in A(V)$.

Now

$$\begin{aligned} S^m &= (\alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k)^m \\ &= T^m (\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1})^m \\ &= 0 \quad [:\because T^m = 0] \end{aligned}$$

Thus S is a nilpotent transformation with $S^m = 0$.

Then by our discussion as above, $\alpha_0 + S$ is invertible for $\alpha_0 \neq 0$.

Hence for $\alpha_0 \neq 0$, $\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k$ is invertible.

Lemma

If $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, where each subspace V_i is of dimension n_i and is invariant under T , an element of $A(V)$, then a basis of V can be found so that the matrix of T in this basis is of the form

$$\begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{bmatrix}$$

where each A_i is an $n_i \times n_i$ matrix and is the matrix of the linear transformation induced by T on V_i .

Proof:

Given $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, where each subspace V_i is of dimension n_i and is invariant under T .

Let $v_{i1}, v_{i2}, \dots, v_{in_i}$ be a basis of V_i ($1 \leq i \leq k$).

Then

$$\left\{ v_{11}, v_{12}, \dots, v_{1n_1}, v_{21}, v_{22}, \dots, v_{2n_2}, \dots, \right. \\ \left. v_{i1}, v_{i2}, \dots, v_{in_i}, \dots, v_{k1}, v_{k2}, \dots, v_{kn_k} \right\}$$

is a basis of V .

Let T_i be the linear transformation induced by T on V_i and let A_i be the matrix of T_i with respect to the basis $v_{i1}, v_{i2}, \dots, v_{in_i}$.

Since each V_i is invariant under T , $V_i T \subset V_i$.

Therefore, for each i , $v_{ij} T$ is a linear combination of the vectors $v_{i1}, v_{i2}, \dots, v_{in_i}$ and only of these.

Therefore, matrix of T in the above basis is

$$\begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_k \end{bmatrix}$$

Theorem

If $T \in A(V)$ is nilpotent with index of nilpotence n_1 then there always exist subspaces V_1 and W invariant under T so that

$$V = V_1 \oplus W.$$

Proof: For proving the theorem, first we prove two lemmas:

Lemma1

If $T \in A(V)$ is nilpotent with index of nilpotence n_1 then there always exists subspace V_1 of V of dimension n_1 which is invariant under T .

Proof of Lemma1: Since $T \in A(V)$ is nilpotent with index of nilpotence n_1 , we have

$$T^{n_1} = 0$$

and

$$T^k \neq 0 \quad \text{for all } k \text{ with } 0 < k \leq n_1 - 1.$$

Since $T^{n_1-1} \neq 0$, there exists $v \neq 0$ in V such that $vT^{n_1-1} \neq 0$.

Consider the elements $v, vT, vT^2, \dots, vT^{n_1-1}$ of V .

Claim: $v, vT, vT^2, \dots, vT^{n_1-1}$ are linearly independent.

Let $\alpha_1 v + \alpha_2 vT + \dots + \alpha_s vT^{s-1} + \dots + \alpha_{n_1} vT^{n_1-1} = 0$ where $\alpha_i \in F$.

Let α_s be the first non-zero element in the above equation. Then

$$\alpha_s vT^{s-1} + \dots + \alpha_{n_1} vT^{n_1-1} = 0.$$

$$\implies vT^{s-1}(\alpha_s + \alpha_{s+1}T + \dots + \alpha_{n_1}T^{n_1-s}) = 0$$

Since T is nilpotent and $\alpha_s \neq 0$, we have

$\alpha_s + \alpha_{s+1}T + \dots + \alpha_{n_1}T^{n_1-s}$ is invertible and so $vT^{s-1} = 0$.

Note that $s - 1 < n_1$.

$vT^{s-1} = 0$ for $s < n_1$.

Which is a contradiction to $vT^{n_1-1} \neq 0$.

Thus $\alpha_i = 0$ for all i and hence $v, vT, vT^2, \dots, vT^{n_1-1}$ are linearly independent over F .

Let V_1 be the space generated by the elements

$v, vT, vT^2, \dots, vT^{n_1-1}$.

That is, $V_1 = \text{Span}(v, vT, vT^2, \dots, vT^{n_1-1})$.

Then $\dim(V_1) = n_1$.

It remains to prove V_1 is invariant under T .

Let $u \in V_1$. Then

$$u = \beta_1 v + \beta_2 vT + \cdots + \beta_{n_1} vT^{n_1-1}.$$

$$\therefore, uT = (\beta_1 v + \beta_2 vT + \cdots + \beta_{n_1} vT^{n_1-1})T$$

$$\implies uT = \beta_1 vT + \beta_2 vT^2 + \cdots + \beta_{n_1-1} vT^{n_1-1} + \beta_{n_1} vT^{n_1}$$

$$\implies uT = \beta_1 vT + \beta_2 vT^2 + \cdots + \beta_{n_1-1} vT^{n_1-1} \text{ since } T^{n_1} = 0.$$

Thus uT is also a linear combination of $v, vT, vT^2, \dots, vT^{n_1-1}$ over F and hence $uT \in V_1$.

Hence V_1 is invariant under T .

Lemma2

If V_1 is subspace of V spanned by $v, vT, vT^2, \dots, vT^{n_1-1}$ where $T \in A(V)$ is nilpotent with index of nilptence n_1 and $u \in V_1$ is such that

$$uT^{n_1-k} = 0, \quad 0 < k < n_1$$

then

$$u = u_0 T^k$$

for some $u_0 \in V_1$.

Proof of lemma2:

Given $u \in V_1$ and $uT^{n_1-k} = 0$, $0 < k < n_1$.

$u \in V_1 \implies$

$u = \alpha_1 v + \alpha_2 vT + \dots + \alpha_k vT^{k-1} + \alpha_{k+1} vT^k + \dots + \alpha_{n_1} vT^{n_1-1}$, $\alpha_i \in F$.

$$\begin{aligned} 0 &= uT^{n_1-k} \\ &= (\alpha_1 v + \alpha_2 vT + \dots + \alpha_k vT^{k-1} + \alpha_{k+1} vT^k \\ &\quad + \dots + \alpha_{n_1} vT^{n_1-1}) T^{n_1-k} \\ &= \alpha_1 vT^{n_1-k} + \alpha_2 vT^{n_1-k+1} + \dots + \alpha_k vT^{k-1+n_1-k} \\ &\quad + \alpha_{k+1} vT^{k+n_1-k} + \dots + \alpha_{n_1} vT^{n_1-1+n_1-k} \end{aligned}$$

$$\begin{aligned} \implies 0 &= \alpha_1 v T^{n_1-k} + \alpha_2 v T^{n_1-k+1} + \dots + \alpha_k v T^{n_1-1} \\ &\quad + \alpha_{k+1} v T^{n_1} + \dots + \alpha_{n_1} v T^{2n_1-1-k} \end{aligned}$$

Since T is nilpotent of index n_1 , $T^{n_1} = 0$.

Therefore,

$$\alpha_1 v T^{n_1-k} + \alpha_2 v T^{n_1-k+1} + \dots + \alpha_k v T^{n_1-1} = 0$$

Since $vT^{n_1-k}, \dots, vT^{n_1-1}$ are linearly independent, we get

$$\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_k = 0.$$

This implies that

$$u = \alpha_{k+1}vT^k + \dots + \alpha_{n_1}vT^{n_1-1}$$

$$\implies u = (\alpha_{k+1}v + \dots + \alpha_{n_1}vT^{n_1-k-1}) T^k$$

Take $u_0 = \alpha_{k+1}v + \dots + \alpha_{n_1}vT^{n_1-k-1}$.

Then $u_0 \in V_1$ and $u = u_0 T^k$.

Hence the lemma.

Proof of Main Theorem:

Given T is nilpotent with index of nilpotence n_1 .

Then by Lemma , there always exist a subspace V_1 of V generated by $v, vT, vT^2, \dots, vT^{n_1-1}$.

Let W be the subspace of V of maximal dimension such that

- $V_1 \cap W = \{0\}$
- W is invariant under T .

We have to show that $V = V_1 + W$.

Suppose not. (i.e) suppose $V \neq V_1 + W$.

Then $\exists z \in V$ such that $z \notin V_1 + W$.

Since $T^{n_1} = 0$, $zT^{n_1} = 0$.

But then there exists an integer k with $0 < k \leq n_1$ such that

$zT^k \in V_1 + W$ and $zT^i \notin V_1 + W$ for $0 < i < k$.

Let $zT^k = u + w$, where $u \in V_1$ and $w \in W$.

Then $zT^{n_1} = 0 \implies (zT^k)T^{n_1-k} = 0$.

$$\implies (u + w)T^{n_1-k} = 0$$

$$\implies uT^{n_1-k} + wT^{n_1-k} = 0$$

$$\implies uT^{n_1-k} = -wT^{n_1-k}$$

Since both V_1 and W are invariant under T , $uT^{n_1-k} \in V_1$ and $wT^{n_1-k} \in W$.

Note that $V_1 \cap W = \{0\}$.

Therefore, $uT^{n_1-k} = -wT^{n_1-k} \implies uT^{n_1-k} = 0$.

By the lemma 2, $u = u_0T^k$ for some $u_0 \in V_1$.

$\therefore, zT^k = u + w \implies zT^k = u_0T^k + w$.

Take $z_1 = z - u_0$.

Then $z_1T^k = (z - u_0)T^k = w$ and so $z_1T^k \in W$.

Further W is invariant under T implies that $z_1 T^m \in W$ for all $m \geq k$.

For $i < k$, $z_1 T^i = zT^i - u_0 T^i$.

Since $zT^i \notin V_1 + W$, we have $z_1 T^i \notin W$.

Therefore, $z_1 T^i \notin W$ for $i < k$.

Let W_1 be the subspace of V spanned by $w, z_1 T, z_1 T^2, \dots, z_1 T^{k-1}$.

Since $z_1 \notin W$, $W \subset W_1$ and $\dim(W) < \dim(W_1)$

Now $z_1 T^k \in W$ and W is invariant under T implies that W_1 is also invariant under T .

Since W is the maximal subspace of V with

- $V_1 \cap W = \{0\}$
- W is invariant under T

we have $V_1 \cap W_1 \neq \{0\}$.

Therefore, there exists an element of the form

$w_0 + a_1 z_1 + a_2 z_1 T + \cdots + a_k z_1 T^{k-1} \neq 0$ in $W_1 \cap V_1$. Here $w_0 \in W$.

This implies that not all the a_1, a_2, \dots, a_k are zero.

Because otherwise $w_0 \neq 0$ be an element in $V_1 \cap W = \{0\}$. Which is not possible.

Let a_s be the first nonzero element. Then

$$w_0 + a_s z_1 T^{s-1} (a_s + a_{s+1} + \dots + a_k T^{k-s}) \in V_1.$$

Since $a_s \neq 0$ and T is nilpotent transformation, we have

$$a_s + a_{s+1} + \dots + a_k T^{k-s}$$

is invertible and its inverse R is a polynomial in T .

Therefore, both V_1 and W are invariant under T .

Note that from the above, $w_0R + z_1T^{s-1} \in V_1R \subset V_1$ and $w_0 \in W$ implies that $z_1T^{s-1} \in V_1 + WR \subset V_1 + W$.

Since $s - 1 < k$, $z_1T^{s-1} \notin V_1 + W$, we get a contradiction here.

Therefore, $V = V_1 + W$.

Now $V_1 \cap W = \{0\}$ implies that $V = V \oplus W$.

Hence the proof.

Notation

Let M_t denote the $t \times t$ matrix of whose entries are zero except the super diagonal. All super diagonal entries are 1's.

$$M_t = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Remark

If V_1 is a subspace of dimension n_1 spanned by $v_1 = v$,
 $v_2 = vT$, $v_3 = vT^2$, \dots , $v_{n_1} = vT^{n_1-1}$, then in this basis the matrix
of T on V_1 is M_{n_1} .

Theorem

If $T \in A(V)$ is nilpotent, of index of nilpotence n_1 , then a basis of V can be found such that the matrix of T in this basis has the form

$$\begin{bmatrix} M_{n_1} & 0 & 0 & \dots & 0 \\ 0 & M_{n_2} & 0 & \dots & 0 \\ 0 & 0 & M_{n_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_{n_r} \end{bmatrix}$$

where $n_1 \geq n_2 \geq \dots \geq n_r$ and $\dim(V) = n = n_1 + n_2 + \dots + n_r$.

Proof:

We prove this theorem in several steps.

Step 1 If $T \in A(V)$ is nilpotent with index of nilpotence n_1 then there always exists subspace V_1 of V of dimension n_1 which is invariant under T .

Since T is nilpotent of index of nilpotence n_1 , $T^{n_1} = 0$ and $T^{n_1-1} \neq 0$.

Therefore, $\exists v \neq 0$ in V such that $vT^{n_1-1} \neq 0$.

We claim that $\{v, vT, vT^2, \dots, vT^{n_1-1}\}$ is linearly independent.

Let $\alpha_1 v + \alpha_2 vT + \dots + \alpha_s vT^{s-1} + \dots + \alpha_{n_1} vT^{n_1-1} = 0$ where $\alpha_i \in F$.

Suppose α_s be the first non-zero element in the above equation.

Then

$$\alpha_s vT^{s-1} + \dots + \alpha_{n_1} vT^{n_1-1} = 0.$$

$$\implies vT^{s-1}(\alpha_s + \alpha_{s+1}T + \dots + \alpha_{n_1}T^{n_1-s}) = 0$$

Since T is nilpotent and $\alpha_s \neq 0$, we have

$\alpha_s + \alpha_{s+1}T + \dots + \alpha_{n_1}T^{n_1-s}$ is invertible and so $vT^{s-1} = 0$.

Here $vT^{s-1} = 0$ for $s < n_1$.

Which is a contradiction to $vT^{n_1-1} \neq 0$.

Thus $\alpha_i = 0$ for all i and hence $v, vT, vT^2, \dots, vT^{n_1-1}$ are linearly independent over F .

Let V_1 be the space generated by the elements $v, vT, vT^2, \dots, vT^{n_1-1}$.

That is, $V_1 = \text{Span}(v, vT, vT^2, \dots, vT^{n_1-1})$.

Then $\dim(V_1) = n_1$.

We prove that V_1 is invariant under T .

Let $u \in V_1$. Then

$$u = \beta_1 v + \beta_2 vT + \cdots + \beta_{n_1} vT^{n_1-1}.$$

$$\therefore, uT = (\beta_1 v + \beta_2 vT + \cdots + \beta_{n_1} vT^{n_1-1})T$$

$$\implies uT = \beta_1 vT + \beta_2 vT^2 + \cdots + \beta_{n_1-1} vT^{n_1-1} + \beta_{n_1} vT^{n_1}$$

$$\implies uT = \beta_1 vT + \beta_2 vT^2 + \cdots + \beta_{n_1-1} vT^{n_1-1} \text{ since } T^{n_1} = 0.$$

Thus uT is also a linear combination of $v, vT, vT^2, \dots, vT^{n_1-1}$ over F and hence $uT \in V_1$.

Hence V_1 is invariant under T .

Step 2: If V_1 is subspace of V spanned by $v, vT, vT^2, \dots, vT^{n_1-1}$ where $T \in A(V)$ is nilpotent with index of nilptence n_1 and $u \in V_1$ is such that

$$uT^{n_1-k} = 0, \quad 0 < k < n_1$$

then

$$u = u_0 T^k$$

for some $u_0 \in V_1$.

Proof of Step 2:

Given $u \in V_1$ and $uT^{n_1-k} = 0$, $0 < k < n_1$.

$u \in V_1 \implies$

$u = \alpha_1 v + \alpha_2 vT + \dots + \alpha_k vT^{k-1} + \alpha_{k+1} vT^k + \dots + \alpha_{n_1} vT^{n_1-1}$, $\alpha_i \in F$.

$$\begin{aligned} 0 &= uT^{n_1-k} \\ &= (\alpha_1 v + \alpha_2 vT + \dots + \alpha_k vT^{k-1} + \alpha_{k+1} vT^k \\ &\quad + \dots + \alpha_{n_1} vT^{n_1-1}) T^{n_1-k} \\ &= \alpha_1 vT^{n_1-k} + \alpha_2 vT^{n_1-k+1} + \dots + \alpha_k vT^{k-1+n_1-k} \\ &\quad + \alpha_{k+1} vT^{k+n_1-k} + \dots + \alpha_{n_1} vT^{n_1-1+n_1-k} \end{aligned}$$

$$\begin{aligned} \implies 0 &= \alpha_1 v T^{n_1-k} + \alpha_2 v T^{n_1-k+1} + \dots + \alpha_k v T^{n_1-1} \\ &\quad + \alpha_{k+1} v T^{n_1} + \dots + \alpha_{n_1} v T^{2n_1-1-k} \end{aligned}$$

Since T is nilpotent of index n_1 , $T^{n_1} = 0$.

Therefore,

$$\alpha_1 v T^{n_1-k} + \alpha_2 v T^{n_1-k+1} + \dots + \alpha_k v T^{n_1-1} = 0$$

Since $vT^{n_1-k}, \dots, vT^{n_1-1}$ are linearly independent, we get

$$\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_k = 0.$$

This implies that

$$u = \alpha_{k+1}vT^k + \dots + \alpha_{n_1}vT^{n_1-1}$$

$$\implies u = (\alpha_{k+1}v + \dots + \alpha_{n_1}vT^{n_1-k-1}) T^k$$

Take $u_0 = \alpha_{k+1}v + \dots + \alpha_{n_1}vT^{n_1-k-1}$.

Then $u_0 \in V_1$ and $u = u_0 T^k$.

Step 3: There always exists subspaces V_1 and W invariant under T so that

$$V = V_1 \oplus W.$$

Given T is nilpotent with index of nilpotence n_1 .

Then by Step 1, there always exist a subspace V_1 of V generated by $v, vT, vT^2, \dots, vT^{n_1-1}$.

Let W be the subspace of V of maximal dimension such that

- $V_1 \cap W = \{0\}$
- W is invariant under T .

We have to show that $V = V_1 + W$.

Suppose not. (i.e) suppose $V \neq V_1 + W$.

Then $\exists z \in V$ such that $z \notin V_1 + W$.

Since $T^{n_1} = 0$, $zT^{n_1} = 0$.

But then there exists an integer k with $0 < k \leq n_1$ such that

$zT^k \in V_1 + W$ and $zT^i \notin V_1 + W$ for $0 < i < k$.

Let $zT^k = u + w$, where $u \in V_1$ and $w \in W$.

Then $zT^{n_1} = 0 \implies (zT^k)T^{n_1-k} = 0$.

$$\implies (u + w)T^{n_1-k} = 0$$

$$\implies uT^{n_1-k} + wT^{n_1-k} = 0$$

$$\implies uT^{n_1-k} = -wT^{n_1-k}$$

Since both V_1 and W are invariant under T , $uT^{n_1-k} \in V_1$ and $wT^{n_1-k} \in W$.

Note that $V_1 \cap W = \{0\}$.

Therefore, $uT^{n_1-k} = -wT^{n_1-k} \implies uT^{n_1-k} = 0$.

By the Step 2, $u = u_0T^k$ for some $u_0 \in V_1$.

$\therefore, zT^k = u + w \implies zT^k = u_0T^k + w$.

Take $z_1 = z - u_0$.

Then $z_1T^k = (z - u_0)T^k = w$ and so $z_1T^k \in W$.

Further W is invariant under T implies that $z_1 T^m \in W$ for all $m \geq k$.

For $i < k$, $z_1 T^i = zT^i - u_0 T^i$.

Since $zT^i \notin V_1 + W$, we have $z_1 T^i \notin W$.

Therefore, $z_1 T^i \notin W$ for $i < k$.

Let W_1 be the subspace of V spanned by $w, z_1 T, z_1 T^2, \dots, z_1 T^{k-1}$.

Since $z_1 \notin W$, $W \subset W_1$ and $\dim(W) < \dim(W_1)$

Now $z_1 T^k \in W$ and W is invariant under T implies that W_1 is also invariant under T .

Since W is the maximal subspace of V with

- $V_1 \cap W = \{0\}$
- W is invariant under T

we have $V_1 \cap W_1 \neq \{0\}$.

Therefore, there exists an element of the form

$w_0 + a_1 z_1 + a_2 z_1 T + \cdots + a_k z_1 T^{k-1} \neq 0$ in $W_1 \cap V_1$. Here $w_0 \in W$.

This implies that not all the a_1, a_2, \dots, a_k are zero.

Because otherwise $w_0 \neq 0$ be an element in $V_1 \cap W = \{0\}$. Which is not possible.

Let a_s be the first nonzero element. Then

$$w_0 + a_s z_1 T^{s-1} (a_s + a_{s+1} + \dots + a_k T^{k-s}) \in V_1.$$

Since $a_s \neq 0$ and T is nilpotent transformation, we have

$$a_s + a_{s+1} + \dots + a_k T^{k-s}$$

is invertible and its inverse R is a polynomial in T .

Therefore, both V_1 and W are invariant under T .

Note that from the above, $w_0R + z_1T^{s-1} \in V_1R \subset V_1$ and $w_0 \in W$ implies that $z_1T^{s-1} \in V_1 + WR \subset V_1 + W$.

Since $s - 1 < k$, $z_1T^{s-1} \notin V_1 + W$, we get a contradiction here.

Therefore, $V = V_1 + W$.

Now $V_1 \cap W = \{0\}$ implies that $V = V \oplus W$.

Step 4 Since $V = V_1 \oplus W$ where V_1 and W are invariant under T and dimension of V_1 is n_1 .

Matrix of T on V_1 with respect to the basis

$\{v_1 = v, v_2 = vT, \dots, v_{n_1} = vT^{n_1-1}\}$ of V_1 is an $n_1 \times n_1$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = M_{n_1}$$

Using the basis $\{v_1, v_2, \dots, v_{n_1}\}$ of V_1 and any basis of W , we can find a basis of V in which the matrix of T is of the form

$$\begin{bmatrix} M_{n_1} & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_2 is the matrix of T_2 , the linear transformation induced by T on W .

Since $T^{n_1} = 0$, for some $n_2 \leq n_1$, $T_2^{n_2} = 0$.

Therefore, T_2 is a nilpotent transformation on W with index of nilpotence n_2 .

Thus by the above argument (Step 1 to 3) we can decompose the vector space W as $V_2 \oplus W_1$ where V_2 is of dimension n_2 and matrix of T on V_2 is M_{n_2} .

By continuing this procedure, we can get subspaces V_1, V_2, \dots, V_r of dimensions n_1, n_2, \dots, n_r respectively, each V_i is invariant under T such that

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_r,$$

$n_1 \geq n_2 \geq \dots \geq n_r$ and $\dim(V) = n = n_1 + n_2 + \dots + n_r$.

By continuing this procedure, we can get a basis of V in which the matrix of T is of the form

$$\begin{bmatrix} M_{n_1} & 0 & 0 & \dots & 0 \\ 0 & M_{n_2} & 0 & \dots & 0 \\ 0 & 0 & M_{n_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_{n_r} \end{bmatrix}$$

where $n_1 \geq n_2 \geq \dots \geq n_r$ and $\dim(V) = n = n_1 + n_2 + \dots + n_r$.

Definition

Let $T \in A(V)$ is nilpotent, of index of nilpotence n_1 . Then there exist subspaces V_1, V_2, \dots, V_r of dimensions n_1, n_2, \dots, n_r respectively, each V_i is invariant under T such that

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_r,$$

$n_1 \geq n_2 \geq \dots \geq n_r$ and $\dim(V) = n = n_1 + n_2 + \dots + n_r$. These integers n_1, n_2, \dots, n_k are called invariants of T .

Definition

If $T \in A(V)$ is nilpotent, the subspace M of V , of dimension m , which is invariant under T , is called cyclic with respect to T if

- 1 $MT^m = \{0\}$ and $MT^{m-1} \neq \{0\}$.
- 2 There is an element $z \in M$ such that $z, zT, zT^2, \dots, zT^{m-1}$ form a basis of M .

Note

Condition 1 is actually implied by condition 2.

Lemma

If M of dimension m is cyclic with respect to T , then the dimension of MT^k is $m - k$ for all $k \leq m$.

Proof: Since M is cyclic with respect to T , there exists an element $z \in M$ such that $z, zT, zT^2, \dots, zT^{m-1}$ form a basis of M .

By taking the image of any basis of M under T^k , we can get a basis of MT^k .

Therefore, the image of $z, zT, zT^2, \dots, zT^{m-1}$ under T^k gives a basis $zT^{k+1}, zT^{k+2}, \dots, zT^{m-1}$ of MT^k for $k \leq m$. Since this basis has $m - k$ elements dimension of MT^k is $m - k$ for all $k \leq m$.

Note

If T is nilpotent then by the above theorem we can find integers $n_1 \geq n_2 \geq \cdots \geq n_r$ and subspaces V_1, V_2, \dots, V_r of V which are invariant under T of dimensions n_1, n_2, \dots, n_r respectively such that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

These subspaces V_1, V_2, \dots, V_r are cyclic with respect to T .

Therefore, if T is nilpotent then we can find integers $n_1 \geq n_2 \geq \dots \geq n_r$ and subspaces V_1, V_2, \dots, V_r of V cyclic with respect to T of dimensions n_1, n_2, \dots, n_r respectively such that

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_r$$

Theorem

If $T \in A(V)$ is a nilpotent then the invariants of T are unique.

Proof: Let if possible there are two sets of invariants n_1, n_2, \dots, n_r and m_1, m_2, \dots, m_s of T .

Then $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ and $V = U_1 \oplus U_2 \oplus \dots \oplus U_s$ where each V_i and U_i are cyclic subspaces of V of dimension n_i and m_i respectively.

Now we show that $r = s$ and $n_i = m_i$.

Suppose that k be the first integer such that $n_k \neq m_k$.

(ie.,) $n_1 = m_1, n_2 = m_2 \cdots n_{k-1} = m_{k-1}$ and $n_k \neq m_k$.

Without loss of generality, suppose that $n_k > m_k$.

Consider

$$T^{m_k}(V) = T^{m_k}(V_1) \oplus T^{m_k}(V_2) \oplus \cdots \oplus T^{m_k}(V_r)$$

and

$$\dim(T^{m_k}(V)) = \dim(T^{m_k}(V_1) \oplus T^{m_k}(V_2) \oplus \cdots \oplus T^{m_k}(V_r)).$$

By the above Lemma, $\dim(T^{m_k}(V_i)) = n_i - m_k$.

Therefore,

$$\dim(T^{m_k}(V)) > (n_1 - m_k) + (n_2 - m_k) + \cdots + (n_{k-1} - m_k) \quad (1)$$

Similarly

$$T^{m_k}(V) = T^{m_k}(U_1) \oplus T^{m_k}(U_2) \oplus \cdots \oplus T^{m_k}(U_s)$$

and

$$\dim(T^{m_k}(V)) = \dim(T^{m_k}(U_1) \oplus T^{m_k}(U_2) \oplus \cdots \oplus T^{m_k}(U_s)).$$

\implies

$$\dim(T^{m_k}(V)) = \dim(T^{m_k}(U_1)) + \dim(T^{m_k}(U_2)) + \cdots + \dim(T^{m_k}(U_s))$$

As $m_j \leq m_k$ for $j \geq k$, we have $T^{m_k}(U_j) = \{0\}$.

Therefore, $\dim(T^{m_k}(U_j)) = 0$ for $j \geq k$.

Hence,

$$\dim(T^{m_k}(V)) = (m_1 - m_k) + (m_2 - m_k) + \cdots + (m_{k-1} - m_k) \quad (2)$$

Since $n_1 = m_1, n_2 = m_2 \cdots n_{k-1} = m_{k-1}$, (2) becomes

$$\dim(T^{m_k}(V)) = (n_1 - m_k) + (n_2 - m_k) + \cdots + (n_{k-1} - m_k).$$

Which is contradicting (1). Hence $n_i = m_i$.

Further, $n_1 + n_2 + \cdots + n_r = \dim(V) = m_1 + m_2 + \cdots + m_s$ and $n_i = m_i$ for all i . $\implies r = s$.

Hence the proof.

Matrix analogous

If $n_1 \geq n_2 \geq \dots \geq n_r$ and $m_1 \geq m_2 \geq \dots \geq m_s$ then the matrices

$$A = \begin{bmatrix} M_{n_1} & 0 & \dots & 0 \\ 0 & M_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{n_r} \end{bmatrix}$$

and

$$B = \begin{bmatrix} M_{m_1} & 0 & \cdots & 0 \\ 0 & M_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{m_s} \end{bmatrix}$$

are similar only if $r = s$ and $n_1 = m_1, n_2 = m_2, \dots, n_r = m_r$.

Note

If the invariants are different then the matrices A and B are not similar.

Theorem

Two Nilpotent linear transformations are similar if and only if they have the same invariants.

Proof: Suppose S and T are similar. i.e., there exists a regular mapping A such that $A^{-1}TA = S$

Let n_1, n_2, \dots, n_r be the invariants of S and m_1, m_2, \dots, m_s are that of T .

Then $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ and $V = U_1 \oplus U_2 \oplus \dots \oplus U_s$ where each V_i and U_i are cyclic and invariant subspace of V of dimension n_i and m_i respectively.

As $S(V_i) \subset V_i$,

$$(A^{-1}TA)(V_i) \subset V_i \Rightarrow (A^{-1}T)A(V_i) \subset V_i.$$

Put $A(V_i) = U_i$ (since A is regular).

Therefore $\dim(V_i) = \dim(U_i) = n_i$.

Further, $T(U_i) = TA(V_i) = AS(V_i)$

As $S(V_i) \subset V_i$, therefore $T(U_i) \subset U_i$.

Equivalently, we have to show that U_i is invariant under T .

Moreover,

$$V = A(V) = A(V_1) \oplus A(V_2) \oplus \cdots \oplus A(V_r) = U_1 \oplus U_2 \oplus \cdots \oplus U_s.$$

By the above theorem, the invariants of Nilpotent transformations are unique. Therefore $n_i = m_i$ and $r = s$.

Conversely, Suppose that two Nilpotent transformations S and T have same invariants.

Then there exists two bases say $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_n\}$ of V such that the matrix of S under $\{v_1, v_2, \cdots, v_n\}$ is equal to the matrix of T under $\{u_1, u_2, \cdots, u_n\}$.

Let it be

$$m(S) = m(T) = \begin{pmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_r} \end{pmatrix}$$

where $m(S) = (\alpha_{ij})$ and $m(T) = (\beta_{ij})$.

Define a linear transformation $A : V \rightarrow V$ by $A(v_i) = u_i$.

$$\begin{aligned}(A^{-1}TA)(v_i) &= (A^{-1}T)A(v_i) \\ &= A^{-1}T(u_i) \\ &= A^{-1}\left(\sum_{j=1}^n \alpha_{ij}u_j\right) \\ &= \sum_{j=1}^n \alpha_{ij}A^{-1}(u_j) \\ &= \sum_{j=1}^n \alpha_{ij}v_j = S(v_i).\end{aligned}$$

Thus $A^{-1}TA = S$. (i.e) S and T are similar. Hence the proof.

Remark

The invariants of T determine a partition of $n = \dim(V)$. Conversely any partition of n such that $n_1 \geq n_2 \geq \cdots, n_r$ and $n = n_1 + n_2 + \cdots + n_r$ determines the invariants of the nilpotent linear transformation.

Thus the number of distinct similarity classes of nilpotent $n \times n$ matrices is precisely $p(n)$, the number of partitions of n

Example

Let

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in F_3$$

act on $F^{(3)}$ with the basis $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (0, 0, 1)$.

Let $v_1 = u_1$, $v_2 = u_1 T = u_2 + u_3$, $v_3 = u_3$ be another basis of F_3 .

Then

$$v_1 T = u_1 T = v_2 = 0.v_1 + 1.v_2 + 0.v_3$$

$$v_2 T = u_2 T + u_3 T = 0.v_1 + 0.v_2 + 0.v_3$$

$$v_3 T = u_3 T = 0.v_1 + 0.v_2 + 0.v_3$$

Matrix of T in this basis v_1, v_2, v_3 is

$$M = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

The invariants of T are 2, 1.

If A is the matrix of change of basis then

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $M = ATA^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$