M304T: Linear Algebra

Canonical Form

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Canonical form

Let V be an n-dimensional vector space over a field F.

Definition

Two linear transformations $S, T \in A(V)$ are said to be similar if $\exists$ an invertible linear transformation $C \in A(V)$ such that

$$T = CSC^{-1}.$$

In this case, we write $T \sim S$.

Definition

Two matrices $A, B \in F_n$ are said to be similar matrices if there exists an invertible matrix $C \in F_n$ such that

$$B = CAC^{-1}.$$

Similarity relation is an equivalence relation on $A(V)$.

- For $T \in A(V),\ T = ITI^{-1}$.

  $\therefore T \sim T$.

- If $T \sim S$ then $\exists$ an invertible element $C \in A(V)$ such that

$$T = CSC^{-1}.$$

Take $D = C^{-1}$. Then $D \in A(V)$ and D is invertible.

Further $DTD^{-1} = C^{-1}TC = C^{-1}CSC^{-1}C = S$.

$\therefore S = DTD^{-1}$ and $D \in A(V)$ is invertible. Thus $S \sim T$.

- If $T \sim S$ and $S \sim U$ then $\exists$ invertible elements C and D in $A(V)$ such that

$$T = CSC^{-1} \quad \text{and} \quad S = DUD^{-1}.$$

  Therefore, $T = CSC^{-1} = CDUD^{-1}C^{-1}$.

  Thus $\exists$ an invertible element $CD \in A(V)$ such that

  $T = CDU(CD)^{-1}$.

  $\implies T \sim U$.

Thus similarity is an equivalence relation on $A(V)$.
The equivalence class of an element is called its similarity class.

- Given two linear transformations, how can we determine whether or not they are similar?
- We shall prove the existence of linear transformations in each similarity class whose matrix, in some basis, is of a particularly nice form called canonical forms.
- For similar linear transformations the canonical form are same.
- So to determine if two linear transformations are similar, we need to compute a particular canonical form for each and check if these are the same.

- There are many possible canonical forms.
- We consider three canonical forms
    - Triangular form
    - Nilpotent transformations
    - Jordan form
    - Rational canonical form

Definition

A square matrix $D = (d_{ij})$ is said to be diagonal if $d_{ij} = 0$ for $i \neq j$.

(i.e) All the entries off the main diagonal are zeros.

Note

The characteristic roots of a diagonal matrix are precisely the

elements on its main diagonal.

A square matrix is said to be diagonalizable if it is similar to a diagonal matrix.

(i.e) A is diagonalizable if $\exists$ a diagonal matrix D such that $A \sim D$.
(i.e)$\exists$ a diagonal matrix D and an inverible matrix P such that

$$A = PDP^{-1}.$$

Definition

A linear transformation $T \in A(V)$ is said to be diagonal if there exists a basis of V in which matrix of T is diagonal.

Definition

A square matrix $A = (a_{ij})$ is said to be lower triangular matrix if all the entries above the main diagonal are zero.

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is a lower triangular matrix.

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is not a lower triangular matrix.

Definition

A square matrix $A = (a_{ij})$ is said to be upper triangular matrix if all the entries below the main diagonal are zero.

Example

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

is an upper triangular matrix.

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is not an upper triangular matrix.

Definition

A matrix is triangular if it is either lower triangular or upper triangular matrix.

(i.e) A matrix is triangular if all the entries above (or below) the main diagonal are 0.

- If A is triangular and no entry on the main diagonal is 0 then A is invertible.
- If A is triangular and an entry on the main diagonal is 0 then A is singular.
- If A is triangular, then its characteristic roots are precisely the elements on its main diagonal.

Definition

A linear transformation T in $A(V)$ is said to be triangular if there exists a basis of V in matrix of T is triangular.

For $T \in A(V)$, the matrix of T in the basis $v_1, v_2, \ldots, v_n$ is triangular if

$$\begin{aligned} v_1 T &= \alpha_{11} v_1 \\ v_2 T &= \alpha_{21} v_1 + \alpha_{22} v_2 \\ &\vdots \\ v_i T &= \alpha_{i1} v_1 + \alpha_{i2} v_2 + \cdots + \alpha_{ii} v_i \\ &\vdots \\ v_n T &= \alpha_{n1} v_1 + \alpha_{n2} v_2 + \cdots + \alpha_{nn} v_n \end{aligned}$$

That is, $v_i T$ is a linear combination only of v_i and its predecessors in the basis.

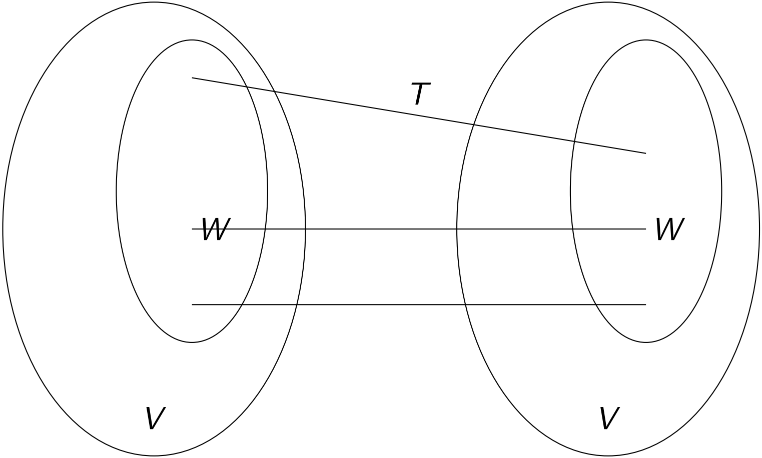
Definition

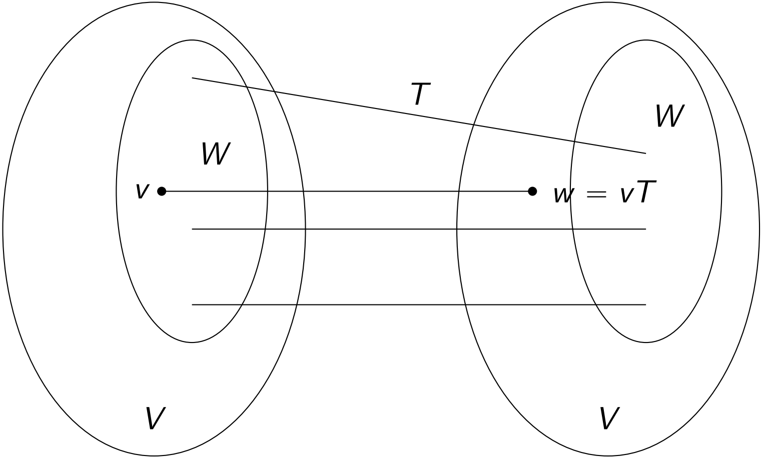
The subspace W of V is invariant under a linear transformation

$T \in A(V)$ if

$$WT \subset W.$$

(ie) image of W under T is in W ($T(W) \subset W$).


T
W
W
V
V



Not invariant because $v \in W$ but $vT = w \in V$ not in W.