PG – 251

First Semester M.Sc. Degree Examination, January/February 2018
(CBCS Scheme)
MATHEMATICS
M 102 T : Real Analysis

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Show that $x^2 \in R\left[x^2\right]$ on [0, 1].

   b) If $f(x) \in R[\alpha(x)]$ on [a, b] then prove that $-f(x) \in R[\alpha(x)]$ on [a, b].

   c) If $f(x) \in R[\alpha(x)]$ on [a, b] and $|f| \le M$, then prove that

   $$\int_a^b |f|\, d\alpha \le M[\alpha(b) - \alpha(a)].$$ **(4+5+5)**

2. a) If $f \in R[\alpha]$ on [a, b] and $c \in R^+$, then prove that $c f \in R[\alpha]$ on [a, b].

   b) If f(x) is continuous on [a, b], $\alpha(x)$ be monotonic on [a, b], then prove that

   $$\int_a^b f d\alpha = f(b)\,\alpha(b) - f(a)\,\alpha(a) - \alpha(\xi)\,[f(b) - f(a)],$$ where $\xi \in (a, b)$.

   c) If $f_1, f_2 \in R[\alpha]$ on [a, b] and $f_1 \le f_2$, then show that $\int_a^b f_1\, d\alpha \le \int_a^b f_2 d\alpha$. **(4+6+4)**

3. a) Let f be Riemann integrable on [a, b] and let $F(x) = \int_a^x f(t)\, dt$, where $a \le x \le b$.

   Then prove that F is continuous on [a, b]. Further, show that f(t) is continuous at a point x_0 on [a, b]. Then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

P.T.O.

b) If $\lim_{\mu(p)\to 0} S(P, f, \alpha)$ exists, then show that $f \in R[\alpha]$ on [a, b] and

$$\lim_{\mu(p)\to 0} S(p, f, \alpha) = \int_a^b f\, d\alpha .$$

c) Given two functions f and g of bounded variation on [a, b]. Show that f + g and f.g are also of bounded variation. **(7+4+3)**

4\. a) Let $\{f_n(x)\}$ be a sequence of functions converges to f(x) defined on [a, b] and $M_n = \sup_{x \in [a,b]} |f_n(x) - f(x)|$. Then prove that $\{f_n(x)\}$ converges to f(x) uniformly on [a, b] if and only if $M_n \to 0$ as $n \to \infty$.

b) Show that $\{e^{-nx}\}$ is uniformly convergent on [a, b].

c) For an infinite series of continuous functions $\sum_{n=1}^{\infty} f_n(x)$ that converges uniformly to f(x) on [a, b], show that f(x) is continuous on [a, b]. **(6+4+4)**

5\. a) If $|f_n(x)| < M_n, \forall n \in N, \forall x \in [a, b]$ and $\sum_{n=1}^{\infty} M_n$ of positive reals, is convergent, then prove that $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on [a, b].

b) Show that $\sum_{n=1}^{\infty} n \times e^{-nx^2}$ converges point-wise and not uniformly on [0, k], k > 0.

c) Let $\sum_{n=0}^{\infty} f_n(x)$ be an infinite series of functions uniformly convergent to f(x) on [a, b] and each $f_n(x) \in R[a, b]$ then prove that $f(x) \in R[a, b]$. Also prove that

$$\int_a^x \left\{ \sum_{n=1}^{\infty} f_n(t) \right\} dt = \sum_{n=k}^{\infty} \left\{ \int_a^x f_n(t)\, dt \right\}.$$

(5+4+5)

6. a) State and prove the Hiene-Borel theorem.

   b) Define a k-cell prove that every k-cell is compact. **(7+7)**

7. a) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and f is differentiable at a point $x \in E$. Then the partial derivatives $(D_j f_i)(x)$ exist, and

$$f'(x)\, e_j = \sum_{i=1}^{m} (D_j f_i)(x)(u_i), (1 \le j \le n).$$

   b) If $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\| T \| < \infty$ and T is uniformly continuous mapping of $\mathbb{R}^n$ onto $\mathbb{R}^m$.

   c) If $\phi : X \to X$ is a contraction on a complete metric space X, then prove that ϕ has a unique fixed point. **(5+4+5)**

8. State and prove the implicit function theorem. **14**
