PG – 255

First Semester M.Sc. Examination, January/February 2018
(CBCS Scheme) (2017 – 18 & Onwards)
MATHEMATICS
M – 107SC : Mathematical Analysis

Time : 3 Hours

Max. Marks : 70

Instructions : *1) Answer* ***any five*** *questions.*
2) ***All*** *questions carry* ***equal*** *marks.*

1. a) Let [x] denote the largest integer contained in x, that is, [x] is the integer such that $x - 1 < [x] \leq x$ x; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions [x] and (x) have ?

   b) Let f be a continuous real function on the interval [a, b]. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then show that there exists a point $x \in (a, b)$ such that $f(x) = c$. (7+7)

2. a) Let f and g be complex continuous functions on a metric space X. Then prove that fg and f/g are continuous on X.

   b) Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then prove that f(X) is compact. (7+7)

3. State and prove the following :

   a) Generalized Mean value theorem.

   b) Taylor's theorem. (7+7)

4. a) Let f be defined on (a, b). If f is discontinuous at a point x, then define what are discontinuities of the first and second kinds.

   b) Plot the following functions :

   i) $$f(x) = \begin{cases} x+2, & -3 < x < -2 \\ -(x+2), & -2 \leq x < 0 \\ x+2, & 0 \leq x < 1 \end{cases}$$

P.T.O.

ii) $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

(Assume sinx is a continuous function)

Classify the discontinuities of the functions into first and second kinds. **(7+7)**

5. a) Suppose the $\{x_n\}$ is monotonic. Then prove that the $\{x_n\}$ converges if and only if it is bounded.

b) Prove that the subsequential limits of a sequence $\{x_n\}$ of real numbers form a closed subset of $\mathbb{R}$.

c) Prove the following :

i) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

ii) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

iii) If $p > 0$ and $\alpha \in R$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$. **(4+4+6)**

6. a) State and Cesaro's limit theorem.

b) Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if for given $\epsilon > 0$, there exists an integer N such that $\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon, \forall m > n \geq N$.

c) Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. **(6+4+4)**

7. a) For any sequence $\{C_n\}$ of positive real numbers, prove that

$$\lim_{n\to\infty} \text{sub} \sqrt[n]{C_n} \leq \lim_{n\to\infty} \text{sub} \frac{C_{n+1}}{C_n}$$

and

$$\lim_{n\to\infty} \inf \sqrt[n]{C_n} \geq \lim_{n\to\infty} \inf \frac{C_{n+1}}{C_n}$$

b) Sate and prove Logarithmic test.

c) If $\sum a_n$ converges and if $\{b_n\}$ is monotonic and bounded, then prove that $\sum a_n b_n$ converges.

(4+6+4)

8. a) State and prove Marten's theorem.

b) If $\sum_{n=1}^{\infty} |a_n|$ converges, then show that every rearrangement of $\sum_{n=1}^{\infty} a_n$ converges and they all converge to the same sum.

c) Assume that each $a_n \geq 0$, $n = 1, 2, 3, \ldots$ then, the product $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if prove that the series $\sum_{n=1}^{\infty} a_n$ converges. (6+4+4)

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I Semester M.Sc. Examination, February 2019
(CBCS Scheme)
MATHEMATICS
M107SC : Mathematical Analysis

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Instructions: *1) Answer **any five** questions.*
*2) **All** questions have **equal** marks.*

1. a) Let (X, d_x) and (Y, d_y) be metric spaces, $E \subset X$ and $p \in E$. The function f is continuous at p, iff for every sequence $\{x_n\}$ in E converging to P, then show that the sequence $\{f(x_n)\}$ in Y converges to f (p).

   b) Show that continuous function on a compact metric space is uniformly continuous. **(7+7)**

2. Define connected metric space. Prove that a continuous mapping from a connected metric space into a metric space is connected. **(14)**

3. a) If f(x) is continuous on [a, b], f′(c) exists at some point $c \in [a, b]$, g(x) is defined on an interval I which contains the range of f(x) and g (x) is differentiable at the point f(c), if h(x) = g(f(x)), then show that h(x) is differentiable at c.

   b) With help of function $f(x) = x^2 \sin(1/x)$, prove the continuity of derivatives. **(7+7)**

4. a) State and prove L' Hospitals rule for indeterminate form of type ∞/∞.

P.T.O.

b) If f (x) is real valued continuous function on [a, b] which is differentiable in (a, b) then show that there exist a point $x \in (a, b)$ such that $f(b) - f(a) = (b - a) f'(x)$. **(7+7)**

5. a) Every bounded sequence of real number contains a convergent subsequence. Prove or disprove it.

b) Prove that a sequence of real number converges iff it is Cauchy sequence. **(7+7)**

6. a) Show that e is irrational.

b) Prove : $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. **(7+7)**

7. a) State and prove Cauchy integral test.

b) Given a series $\sum_{n=1}^{\infty} a_n$

Let $\alpha = \lim_{n \to \infty} \sup \sqrt[\frac{1}{n}]{|a_n|}$

then prove that

i) $\sum_{n=1}^{\infty} a_n$ converges if $\alpha < 1$

ii) $\sum_{n=1}^{\infty} a_n$ diverges if $\alpha > 1$

iii) if $\alpha = 1$, the test gives no information.

c) Test the convergence of

$\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n-1}\right)$ **(5+5+4)**

8. a) State and prove Martin's theorem.

b) Suppose

i) $\sum_{n=0}^{\infty} a_n$ converges absolutely

ii) $\sum_{n=0}^{\infty} a_n = A$

iii) $\sum_{n=0}^{\infty} b_n = B$

iv) $C_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n = 0, 1, 2 \ldots$

then show that

$$\sum_{n=0}^{\infty} C_n = AB$$

c) Assume that each $a_n \geq 0$, $n = 1, 2, \ldots$, then the product $\prod_{n=1}^{\infty} (1 + a_n)$ converges. Iff prove that the series $\sum_{n=1}^{\infty} a_n$ converges. **(5+5+4)**