PG – 143

I Semester M.Sc. Degree Examination, January 2016
(CBCS)
MATHEMATICS
M 102 T : Real Analysis

Time : 3 Hours

Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Show that $(3x + 1)$ is Riemann integrable on [1, 2]. 4

b) Prove that $f \in R[\alpha]$ on [a, b] if and only if given $\varepsilon > 0$, there exists a partition P of [a, b] such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$. 5

c) If $f \in R[\alpha_1]$ on [a, b] and $f \in R[\alpha_2]$ on [a, b], then prove that $f \in R[\alpha_1 + \alpha_2]$ on [a, b]. 5

2. a) If $f_1, f_2 \in R[\alpha]$ on [a, b] and $f_1 \le f_2$, then show that $\int_a^b f_1 \, d\alpha \le \int_a^b f_2 d\alpha$. 5

b) If f(x) is continuous on [a, b] and $\alpha(x)$ be monotonic on [a, b], prove that

$$\int_a^b f.\, d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(\xi)[f(b) - f(a)], \text{ where } \xi \in (a,b).$$ 6

c) Give an example of a function f such that $|f| \in R[\alpha]$ on [0, 1] and $f \notin R[\alpha]$ on [0, 1]. 3

3. a) If $f \in R$ [a, b] and if there exists a function F on [a, b] such that $F' = f$, then prove that $\int_a^b f dx = F(b) - F(a)$. 4

b) If f and ϕ are continuous on [a, b] and ϕ is strictly increasing on [a, b] and ψ is an inverse function of ϕ, then prove that $\int_a^b f(x)dx = \int_{\phi(a)}^{\phi(b)} f(\psi(g))\, d\psi(g)$. 5

c) Define function of bounded variation let f and g be function of bounded variation on [a, b]. Show that $f \pm g$ and f.g are also functions of bounded variation. 5

4. a) Define uniform convergence of a sequence of functions $\{f_n(x)\}$ on [a, b]. State and prove Cauchy's criterion for uniform convergence of $f_n(x)$ on [a, b]. **6**

b) Show that for $-1 < x < 1$, the series $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \ldots = \frac{1}{1-x}$. **4**

c) Test for uniform convergence of the sequence $[\tan^{-1}(nx)]$ on [a, b]. **4**

5. a) Suppose $f_n \to f$ uniformly on [a, b] and if $x_0 \in [a, b]$ such that $\lim\limits_{x \to x_0} f(x) = a_n$ for n = 1, 2, 3, ... Then prove that

i) $\{a_n\}$ converges

ii) $\lim\limits_{x \to x_0} f_n(x) = \lim\limits_{n \to \infty} \lim\limits_{x \to x_0} f_n(x)$. **5**

b) Let $\{f_n(x)\}$ be a sequence of functions uniformly convergent to f(x) on [a, b] and each $f_n(x) \in R[a, b]$. Then prove that $f(x) \in R[a, b]$. Also prove

$$\lim_{n \to \infty} \int_a^x f_n(t)dt = \int_a^x f(t)dt \ \forall x \in [a, b].$$ **5**

c) Show that $\sum\limits_{n=1}^{\infty} nxe^{-nx^2}$ converges point-wise and not uniformly on [0.4], k > 0. **4**

6. a) If A is a sub-set of $\mathbb{R}$. Then prove that the following statements are equivalent.

i) A is closed and bounded

ii) A is compact

iii) A is countably compact. **8**

b) Prove that any infinite bounded subset of $\mathbb{R}^k$ has a limit point in R^k. **6**

7. a) Let 'E' be an open subset of R^n and $f : E \to R^n$ be a differentiable function at $x_0 \in E$. Then prove that f is continuous at x_0 and $f'(x_0)$ is unique. **6**

b) Let 'E' be an open subset of R^n and $f : E \to R^n$ be differentiable at a point $x_0 \in E$. Let F be an open subset of R^n containing E and $g : F \to R^k$ be differentiable at $f(x_0)$. If $\phi = g \circ f : E \to R^k$, then prove that ϕ is differentiable at $x_0 \in E$ and $\phi'(x_0) = g'(f(x_0)) \circ f'(x_0)$. **6**

c) Let $T : R^n \to R^m$ be a mapping with $T = (T_1, T_2, \ldots, T_m)$. Prove that 'T' is linear transformation if and only if $T_i (i = 1, 2, \ldots, m)$ are linear transformations. **2**

8. State and prove the inverse function theorem. **14**

PG – 382

I Semester M.Sc. Examination, January 2017
(CBCS)
MATHEMATICS
M102T : Real Analysis

Time : 3 Hours   Max. Marks : 70

Instructions *: 1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Evaluate $\int_0^x x \, d\{[x]\}$ where [x] is the maximum integer function. **4**

b) If $f \in R[\alpha]$ on [a, b], then prove that $\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha = \int_a^b f \, d\alpha = \lambda\,[\alpha(b) = \alpha(a)]$,

where $\lambda \in [m, M]$. **5**

c) If P* is a refinement of partition P of [a, b], then show that
$L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$. **5**

2. a) Assuming f(x) is monotonic on [a, b] and $\alpha(x)$ is monotonically increasing and continuous functions on [a, b], prove that $f \in R[\alpha]$ on [a, b]. **5**

b) If $f \in R[\alpha]$ on [a, b], $f \in [m, M]$ and ϕ is continuous function of f on [m, M] then prove that $\phi(f(x)) \in R[\alpha]$ on [a, b]. **7**

c) Evaluate $\int_0^5 x^2 d\{[x] - x\}$. **2**

3. a) Consider the functions $\beta_1(x)$ and $\beta_2(x)$ defined as follows :

$$\beta_1(x) = \begin{cases} 0 & \text{when } x \leq 0 \\ 1 & \text{when } x > 0 \end{cases}$$

$$\beta_2(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

verify whether $\beta_1(x) \in R[\beta_2(x)]$ on [–1, 1]. **7**

b) State and prove fundamental theorem of integral calculus. **3**

c) Calculate the total variation function of $f(x) = x - [x]$ on $[0, 2]$, where $[x]$ is the minimum integral function. **4**

4. a) Define uniform convergence of sequences and series of functions. State Weirstrauss M-test for uniform convergence for infinite series

$$\sum_{n=1}^{\infty} f_n(x) \text{ on } [a, b].$$ **7**

b) Test for uniform convergence of the following : **7**

i) $\left\{\dfrac{nx}{1+n^2x^2}\right\}$ for $x \in [0, 1]$.

ii) $\left\{nxe^{-nx^2}\right\}$ for any real x.

iii) $\sum_{n=0}^{\infty} (1-x)\, x^n$ for $x \in [0, 1]$.

5. a) If $\{f_n(x)\}$ is uniformly convergent to $f(x)$ on $[a, b]$ and each $f_n(x)$ is continuous on $[a, b]$ then prove that $f(x)$ is continuous on $[a, b]$. **7**

b) Let $\{f_n(x)\}$ be a sequence of functions uniformly convergent to $f(x)$ on $[a, b]$ and each $f_n(x) \in R(a, b]$. Prove the following :

i) $f(x) \in R[a, b]$,

ii) $\int_a^x \underset{n\to\infty}{\mathrm{Lt}} f_n(t)\, dt = \underset{n\to\infty}{\mathrm{Lt}} \int_a^x f_n(t)\, dt$. **7**

6. a) Define a k-cell in $\mathbb{R}^K$. Let $I_1 \supset I_2 \supset I_3 \supset \ldots$ be a sequence of k-cells in $\mathbb{R}^K$.

Show that $\bigcap_{n=1}^{\infty} I_n \neq \phi$. **7**

b) State and prove Heine-Borel theorem. **7**

7. a) Let $E \subset \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}^m$ is a map. Prove that if f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$. 7

b) If $T_1, T_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$, then prove that

i) $\| T_1 + T_2 \| \leq \| T_1 \| + \| T_2 \|$

ii) $\| \alpha T_1 \| = | \alpha | \| T_1 \|$. 4

c) Let $f : [a, b] \to \mathbb{R}^k$, $f = (f_1, f_2 \ldots, f_k)$, f is differentiable iff each f_i is differentiable. 3

8. State and prove the implicit function theorem. 14

PG – 251

First Semester M.Sc. Degree Examination, January/February 2018
(CBCS Scheme)
MATHEMATICS
M 102 T : Real Analysis

Time : 3 Hours

Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Show that $x^2 \in R\left[x^2\right]$ on [0, 1].

   b) If $f(x) \in R\,[\alpha(x)]$ on [a, b] then prove that $-f(x) \in R\,[\alpha(x)]$ on [a, b].

   c) If $f(x) \in R\,[\alpha(x)]$ on [a, b] and $|f| \le M$, then prove that

   $$\int_a^b |f|\, d\alpha \le M\,[\alpha(b) - \alpha(a)].$$ **(4+5+5)**

2. a) If $f \in R\,[\alpha]$ on [a, b] and $c \in R^+$, then prove that $c\,f \in R[\alpha]$ on [a, b].

   b) If f(x) is continuous on [a, b], $\alpha(x)$ be monotonic on [a, b], then prove that

   $$\int_a^b f d\alpha = f(b)\,\alpha(b) - f(a)\,\alpha(a) - \alpha(\xi)\,[f(b) - f(a)],$$ where $\xi \in (a, b)$.

   c) If $f_1, f_2 \in R[\alpha]$ on [a, b] and $f_1 \le f_2$, then show that $\int_a^b f_1\, d\alpha \le \int_a^b f_2 d\alpha$. **(4+6+4)**

3. a) Let f be Riemann integrable on [a, b] and let $F(x) = \int_a^x f(t)\, dt$, where $a \le x \le b$.

   Then prove that F is continuous on [a, b]. Further, show that f(t) is continuous at a point x_0 on [a, b]. Then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

P.T.O.

b) If $\lim_{\mu(p)\to 0} S(P, f, \alpha)$ exists, then show that $f \in R[\alpha]$ on [a, b] and

$$\lim_{\mu(p)\to 0} S(p, f, \alpha) = \int_a^b f\, d\alpha.$$

c) Given two functions f and g of bounded variation on [a, b]. Show that f + g and f.g are also of bounded variation. (7+4+3)

4. a) Let $\{f_n(x)\}$ be a sequence of functions converges to f(x) defined on [a, b] and $M_n = \sup_{x \in [a,b]} |f_n(x) - f(x)|$. Then prove that $\{f_n(x)\}$ converges to f(x) uniformly on [a, b] if and only if $M_n \to 0$ as $n \to \infty$.

b) Show that $\{e^{-nx}\}$ is uniformly convergent on [a, b].

c) For an infinite series of continuous functions $\sum_{n=1}^{\infty} f_n(x)$ that converges uniformly to f(x) on [a, b], show that f(x) is continuous on [a, b]. (6+4+4)

5. a) If $|f_n(x)| < M_n, \forall n \in N, \forall x \in [a, b]$ and $\sum_{n=1}^{\infty} M_n$ of positive reals, is convergent, then prove that $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on [a, b].

b) Show that $\sum_{n=1}^{\infty} n \times e^{-nx^2}$ converges point-wise and not uniformly on [0, k], k > 0.

c) Let $\sum_{n=0}^{\infty} f_n(x)$ be an infinite series of functions uniformly convergent to f(x) on [a, b] and each $f_n(x) \in R[a, b]$ then prove that $f(x) \in R[a, b]$. Also prove that

$$\int_a^x \left\{ \sum_{n=1}^{\infty} f_n(t) \right\} dt = \sum_{n=k}^{\infty} \left\{ \int_a^x f_n(t)\, dt \right\}.$$

(5+4+

6. a) State and prove the Hiene-Borel theorem.

b) Define a k-cell prove that every k-cell is compact. (7+7)

7. a) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and f is differentiable at a point $x \in E$. Then the partial derivatives $(D_j f_i)(x)$ exist, and

$$f'(x)\, e_j = \sum_{i=1}^{m} (D_j f_i)(x)\,(u_i), \; (1 \leq j \leq n).$$

b) If $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\| T \| < \infty$ and T is uniformly continuous mapping of $\mathbb{R}^n$ onto $\mathbb{R}^m$.

c) If $\phi : X \to X$ is a contraction on a complete metric space X, then prove that ϕ has a unique fixed point. (5+4+5)

8. State and prove the implicit function theorem. 14
