I Semester M.Sc. Degree Examination, Jan. 2016
(CBCS)
MATHEMATICS
M101T : Algebra – I

Time : 3 Hours Max. Marks : 70

*Instructions : 1) Answer **any 5** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define :
   i) Symmetric group S_n.
   ii) Alternating group A_n.

   Show that $S_n / A_n \simeq \{1, -1\}$.

   b) Show that $T : G \to G$ defined by $T(x) = x^{-1}$ is an automorphism if and only if G is abelian.

   c) Show that for every group is isomorphic to a subgroup of A(S) for some appropriate S. **(5+4+5)**

2. a) Let G be a finite group and S is a finite G-set. If $x \in s$ then show that $o(G_x) = o(G)/o(\text{stab }(x))$.

   b) By using generator-relator form of S_3. Verify the class equation of S_3, where S_3 is a symmetric group.

   c) If $o(G) = p^n$, where p is a prime number, prove that $Z(G) \neq \{e\}$, where 'e' is an identity of G. Deduce that every group of order p^2 is abelian. **(4+4+6)**

3. a) Show that the number of p-sylow subgroups of G, for a given prime, is congruent to 1 modulo p.

   b) Let G be a group of order pq, where p and q are primes with p < q and $q \equiv 1 \pmod{p}$. Then show that G is non-abelian.

   c) Show that every group of order $11^2.13^2$ is abelian. **(6+4+4)**

4. a) Define a simple group. Show that a group of order 28 is solvable but not simple.

   b) State and prove the Jordan-Hölder theorem. **(4+10)**

P.T.O.

5. a) If U is an ideal of a ring R, let $[R:U] = \{x \in R : r\,x \in U \ \forall\, r \in R\}$. Prove that $[R : U]$ is an ideal of R containing U.

   b) Show that the homomorphism ϕ of R onto R' is an isomorphic if and only if $\text{Ker}\ \phi = \{0\}$.

   c) State and prove the fundamental theorem of homomorphism for rings. **(4+4+6)**

6. a) Show that a ring $\mathbb{Z}$ of integers is a principle ideal ring.

   b) Define a maximal ideal of a ring R. If R is a commutative ring with unity and M is an ideal of R, then show that M is a maximal ideal of R if and only if R/M is a field.

   c) Show that the quotient field is the smallest field containing D, where D is an integral domain. **(4+6+4)**

7. a) Define an euclidean ring. Let $x = a + ib$, $y = c + id$ be any two elements in $z[i] - \{0\}$. then prove that it is an euclidean ring.

   b) Let R be an euclidean ring. Show that any ideal $A = (a_0)$ is maximal ideal in R if and only if a_0 is a prime element of R.

   c) If p is a prime number of the form $4n + 1$, then show that $x^2 \equiv -1 \pmod{p}$. **(5+5+4)**

8. a) Prove that $\deg(fg) = \deg(f) + \deg(g)$ for $f, g \in R[x]$.

   Further, if R is an integral domain, then show that R[x] is also an integral domain.

   b) State and prove the Euclid's algorithm for polynomials over a field.

   c) Let $A = (x^2 + x + 1)$ be an ideal generated by $x^2 + x + 1 \in Z_2[x]$. Verify that A is a maximal ideal in $Z_2[x]$. **(5+5+4)**

I Semester M.Sc. Degree Examination, January 2017
(CBCS)
Mathematics
M101T : ALGEBRA – I

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any 5** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Let $\phi : G \to G'$ be a homomorphism with Kernel K and let $\overline{N}$ be a normal subgroup of $\overline{G}$ and $N = \{ g \in G : \phi(g) \in \overline{N} \}$. Prove that $G|N \cong \overline{G}|\overline{N}$.

   b) Prove that $I(G) \cong G/Z(G)$, where I (G) is a group of inner automorphisms of G and Z (G) is the centre of G.

   c) Compute the group Aut (K_4), where K_4 is the Klein's 4-group. Hence illustrate that the automorphism group of an abelian group need not be abelian. **(5+4+5)**

2. a) State and prove the Cauchy-Frobenius Lemma.

   b) Derive the class equation for finite groups.

   c) Prove that every group of order p^2, for a prime p is abelian. **(5+5+4)**

3. a) Show that all p-sylow subgroups of a finite group are conjugate to each other.

   b) Show that the number of p-sylow subgroup of n_p of G is of the form $n_p \equiv 1$ (modp).

   c) Show that every group of order 15 is cyclic. **(6+6+2)**

4. a) Show that a normal subgroup N of G is maximal if and only if the quotient group $G|N$ is simple.

   b) If a group G has a composition series, then show that all its composition series are pairwise equivalent.

   c) Define a solvable group. Show that symmetric group S_4 is solvable, but not simple. **(5+6+3)**

5. a) Define integral domain and a field P. Prove that every finite integral domain is a field.

b) Let R be a commutative ring with unity whose ideals are {0} and R only. Prove that R is a field.

c) Let U be the left ideal of a ring R and $\lambda(U) = \{x \in R : xu = 0 \text{ for all } u \in U\}$. Prove that $\lambda(U)$ is an ideal of R. **(6+4+4)**

6. a) Define principal ideal of a ring R. Show that the ring Z of all integers is a principal ideal ring.

b) Let R be an integral domain with ideal P. Show that P is a principal ideal of R if and only if $R|p$ is an integral domain.

c) Show that any two isomorphic integral domains have isomorphic quotient fields. **(4+5+5)**

7. a) Show that every field is an Euclidean ring.

b) Let R be an Euclidean ring and a, b $\in$ R be non-zero with 'b' non-unit. Then prove that d (a) < d (ab).

c) If p is a prime number of the form 4n + 1, prove that $p = a^2 + b^2$ for some integers 'a' and 'b'. **(4+4+6)**

8. a) If F is a field, then show that F [x] is not a field.

b) State and prove Eisenstein criterion for irreducibility of a polynomial.

c) Let $A = (x^2 + x + 1)$ be an ideal generated by $x^2 + x + 1 \in z_2[x]$. Verify that A is a maximal ideal in $z_2[x]$. **(4+5+5)**

PG – 250

I Semester M.Sc. Degree Examination, January/February 2018
(CBCS Scheme)
MATHEMATICS
M101T : Algebra – I

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any 5** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Let $\phi : G \to \overline{G}$ be an epimorphism with Kernel K and let N be a normal subgroup of G. Then prove that $\frac{G/K}{N/K} \approx G/N$.

   b) Show that $T : G \to G$ defined by $T(x) = x^{-1}$ is an automorphism if and only if G is abelian.

   c) State and prove the Cayley's theorem for finite groups. **(5+4+5)**

2. a) State and prove the orbit-stabilizer theorem.

   b) Derive the class equation for finite groups.

   c) Define a p-group. If G is a finite group of prime power order. Prove that G has a non-trivial center. **(5+5+4)**

3. a) State and prove the Sylow first theorem.

   b) Let $O(G) = pq$, where p and q are distinct primes with $p < q$ and $q \not\equiv 1 \pmod{p}$. Then prove that G is abelian and cyclic. **(8+6)**

4. a) Define a solvable group. Prove that every subgroup of a solvable group is solvable.

   b) State and prove the Jordan-Holder theorem.

   c) Show that symmetric group S_4 is solvable, but not solvable. **(4+7+3)**

P.T.O.

5. a) If R is a ring with unity in which (0) and R are the only two left ideals, then prove that R is a divison ring.

   b) If U is an ideal of a ring R, let $[R : U] = \{x \in R : rx \in U \ \forall \ r \in R\}$. Prove that $[R : U]$ is an ideal of R containing U.

   c) Let R and R′ be rings and ϕ is a homomorphism of R onto R′ with Kernel U. Then show that $R' \approx R/U$. **(5+4+5)**

6. a) Define a principal ideal and principal ideal ring. Prove that every field is a principal ideal ring.

   b) Define maximal ideal of a ring. If R is a commutative ring with unit element and M is an ideal of R, then show that M is a maximal ideal of R if and only if R/M is a field.

   c) Prove that in a principal ideal ring, every non-zero prime ideal is maximal ideal. **(5+6+3)**

7. a) Define an euclidean ring. Let $x = a + ib$, $y = c + id$ be any two elements in $Z[i] - \{0\}$ then prove that it is an euclidean ring.

   b) Show that every Euclidean ring is a principle ideal ring.

   c) State and prove the unique factorization theorem. **(5+4+5)**

8. a) Prove that $\deg(fg) = \deg(f) + \deg(g)$ for f, G $\in$ R[x]. Further, if R is an integral domain, then show that R[x] is also an integral domain.

   b) Show that the product of two primitive polynomials is a primitive polynomial.

   c) Verify that $f(x) = x^3 + x^2 - 2x - 1 \in Q[x]$ is irreducible polynomial, by using Eisenstein criteria. **(5+5+4)**