PG – 607

I Semester M.Sc. Degree Examination, January 2015
(CBCS Scheme)
MATHEMATICS
M-102T : Real Analysis

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer* ***any five*** *questions.*
2) ***All*** *questions carry* ***equal*** *marks.*

1. a) Show that f (x) = – x $\in$ R [– c, 0]. **4**

b) If $f(x) \in R\left[\alpha(x)\right]$ on [a, b] then prove that $-f(x) \in R\left[\alpha(x)\right]$ on [a, b]. **5**

c) If $f(x) \in R\left[\alpha(x)\right]$ on [a, b] and $|f| \leq M$, then prove that

$\int_a^b |f| \, d\alpha \leq M\left[\alpha(b) - \alpha(a)\right]$. **5**

2. a) If $f \in R[\alpha]$ on [a, b] and $C \in R^+$, then prove that $Cf \in [\alpha]$ on [a, b]. **4**

b) If f is continuous on [a, b] and α is monotonically increasing function on [a, b], show that $f \in R[\alpha]$. **4**

c) Let f be Riemann integrable on [a, b] and let $F(x) = \int_a^x f(t)dt$, where $a \leq x \leq b$. Then prove that F is continuous on [a, b]. Further, show that if f (t) is continuous at a point x_0 on [a, b], then F is differentiable at x_0 and $F'(x_0) = f(x_0)$. **6**

P.T.O.

3. a) Consider two functions $\beta_1(x)$ and $\beta_2(x)$ defined as follows :

$$\beta_1(x) = \begin{cases} 0 & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$$

$$\beta_2(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \ge 0 \end{cases}$$

Verify whether $\beta_1(x) \in R\left[\beta_2(x)\right]$ on [–1, 1]. **7**

b) If $\lim\limits_{\mu(p)\to 0} S(P, f, \alpha)$ exists, then show that $f \in R[\alpha]$ on [a, b] and

$$\lim_{\mu(p)\to 0} S(p, f, \alpha) = \int_a^b f d\alpha .$$ **3**

c) Calculate the total variation functions of f (x) = x – [x] on [0, 2] where [x] is the maximum integral function. **4**

4. a) State and prove Weier strauss M-test. **5**

b) Test for uniform convergence for $\left\{\frac{nx}{1+n^2x^2}\right\}$ on [0, 1]. **4**

c) Suppose $f_n \to f$ uniformly on [a, b] and if $x_0 \in$ [a, b] such that $\lim\limits_{x\to x_0} f_n(x) = A_n,\ n = 1, 2, 3...$ then prove that

i) A_n converges

ii) $\lim\limits_{x\to x_0} \underset{n\to\infty}{\text{lt}}\ f_n(x) = \underset{n\to\infty}{\text{lt}}\ \lim\limits_{x\to x_0} f_n(x)$. **5**

5. a) If $|f_n(x)| < M_n,\ \forall n \in N,\ \forall x \in [a, b]$ and $\sum\limits_{n=1}^{\infty} M_n$ of positive reals, is convergent, then prove that $\sum\limits_{n=1}^{\infty} f_n(x)$ is uniformly convergent on [a, b]. **5**

b) Show that $\sum_{n=1}^{\infty} nxe^{-nx^2}$ converges point-wise and not uniformly on [0, 4] k > 0. **4**

c) Let $\sum_{n=0}^{\infty} f_n(x)$ be an infinite series of functions uniformly convergent to f(x) on [a, b] and each $f_n(x) \in R[a, b]$ then prove that $f(x) \in R[a, b]$. Also prove that

$$\int_a^x \left\{ \sum_{n=1}^{\infty} f_n(t) \right\} dt = \sum_{n=k}^{\infty} \left\{ \int_a^x f_n(t)dt \right\}.$$ **5**

6. a) State and prove Stone-Weierstrass theorem. **8**

b) Define a k-cell in R^k prove that every k-cell is compact in R^k. **6**

7. a) Let $E \subset \mathbb{R}^n$ be an open set and $f : E \to R^m$ is a map. Prove that if f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \le i \le m$, $1 \le j \le n$. **7**

b) Let E be an open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a differentiable function at $x_0 \in E$. Then prove that f is continuous at x_0 and $f'(x_0)$ is unique. **5**

c) Discuss the continuity of the function f (x, y) = $\begin{cases} \frac{xy}{x^2+y^2}, & x \neq 0 \quad y \neq 0 \\ 0 & x = 0 \quad y = 0 \end{cases}$. **2**

8. State and prove the implicit function theorem. **14**
