PG – 371

First Semester M.Sc. Degree Examination, Jan./Feb. 2014
(RNS) (Y2K11 Scheme)
MATHEMATICS
M – 102 : Real Analysis

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer **any five** questions choosing at least **one** from **each** Part.*
*2) **All** questions carry **equal** marks.*

PART – A

1. a) Show that $(3x + 1)$ is Riemann integrable on [1, 2]. **4**

b) Prove that $f \in R[\alpha]$ on [a, b] if and only if given $\in > 0$, there exists a partition P of [a, b] such that $U(P, f, \alpha) - L(P, f, \alpha) < \in$. **6**

c) If $f \in R[\alpha]$ on [a, b] then prove that $-f \in R[\alpha]$ on [a, b]. **6**

2. a) Let F be Riemann integrable on [a, b] and let $F(x) = \int_a^x f(t)\,dt$, where $a \le x \le b$, then prove that F is continuous on [a, b]. Further, show that if F(t) is continuous at a point x_0 in [a, b], then F is differentiable at x_0 and $F'(x_0) = F(x_0)$. **8**

b) If f(x) is continuous on [a, b] and α (x) be monotonic on [a, b]. Prove that

$\int_a^b f\,d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(\xi)[f(b) - f(a)]$ where $\xi \in (a, b)$. **6**

c) Give an example of a function f such that $|f| \in R[\alpha]$ on [0, 1] and $f \notin R[\alpha]$ on [0, 1]. **2**

P.T.O.

3. a) If $f \in R[a, b]$ and If there exists a function F on [a, b] such that $F' = f$ then prove that $\int_a^b f\,dx = F(b) - F(a)$. **4**

b) If f and ϕ are continuous on [a, b], and ϕ is strictly increasing on [a, b] and ψ is an inverse function of ϕ, then prove that $\int_a^b f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(\psi(y))\,d\psi(y)$. **6**

c) Define functions of bounded variation let f and g be functions of bounded variation on [a, b]. Show that $f \pm g$ and fg are also functions of bounded variations. **6**

PART – B

4. a) Define uniform convergence of a sequence of functions $\{f_n(x)\}$ on [a, b]. State and prove Cauchy's criterion for uniform convergence of $f_n(x)$ on [a, b]. **8**

b) Show that for $-1 < x < 1$, the series $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \ldots = \frac{1}{1-x}$. **4**

c) Test for uniform convergence of the sequence $\{\tan^{-1}(nx)\}$ on [a, b]. **4**

5. a) Suppose $f_n \longrightarrow f$ uniformly on [a, b] and if $x_0 \in [a, b]$ such that $\underset{x \to x_0}{\text{lt}} f_n(x) = a_n$ for n = 1, 2, 3 . . . then prove that

i) $\{a_n\}$ converges

ii) $\lim_{x \to x_0} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x)$. **6**

b) Show that $\sum_{n=1}^{\infty} nxe^{-nx^2}$ converges pointwise and not uniformly on [0, k], k > 0. **5**

c) Let $\sum_{n=0}^{\infty} f_n(x)$ be an infinite series of functions uniformly convergent to f(x) on [a, b] and each $f_n(x) \in R[a, b]$ then prove that $f(x) \in R[a, b]$. Also prove that

$$\int_a^x \left\{ \sum_{n=1}^{\infty} f_n(t) \right\} dt = \sum_{n=k}^{\infty} \left\{ \int_a^x f_n(t)\,dt \right\}.$$ **5**

6. a) If A is a sub-set of IR, then prove that the following statements are equivalent :
   i) A is closed and bounded
   ii) A is compact. **8**

   b) State and prove Bolzano-Weierstrass theorem. **8**

PART – C

7. a) Let $E \subset R^n$ be an open set and $f : E \to R^m$ is a map. Prove that f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \leq i \leq m, 1 \leq j \leq n$. **8**

   b) If $\phi : X \to X$ is a contraction on a complete metric space X, then prove that ϕ has a unique fixed point. **5**

   c) Let $f : [a, b] \to R^k$, $f = (f_1, f_2, \ldots f_k)$, f is differentiable iff each f_i is differentiable. **3**

8. State and prove inverse function theorem. **16**